QNo:→ Prove that, $\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{\pi(b+2}{2bc^3(b+c}$

$(b>0, c>0$

Soln:→ $\int_C \frac{dz}{(z^2+b^2)(z^2+c^2)^2} = \int_C f(z)\,dz.$

Where C is the Contour Consisting of the real lucius $-R$ to R together with the sum circle Γ of radius R in the upper half of the Z plane. The poles of $f(z)$ are given by $(z^2+b^2)(z^2+c^2)^2 = 0$

i.e. $z^2+b^2=0,\ z^2+c^2=0$ (twice)

i.e. $z = \pm bi,\ z = \pm ci$ (twice)

of which only $z = bi$ and $z = ci$ (twice) lies within C.

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz.$$

Since, $\lim_{z\to 0} z f(z)$

$$= \lim_{z\to\infty} z \cdot \frac{1}{(z^2+b^2(z^2+c^2)^2} = 0$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+.$$

where $\sum R^+$ is the Some of the

~~raditure~~ residue at the Poles of $f(z)$ within C.

The residue of $f(z)$ at $z = bi$

$$= \lim_{z \to bi} (z - bi) f(z)$$

$$= \lim_{z \to bi} \frac{(z-bi)}{(z^2+b^2)(z^2+c^2)^2}$$

$$= \lim_{z \to bi} \frac{\cancel{z-bi}}{(z+bi)\cancel{(z-bi)}(z^2+c^2)^2}$$

$$= \frac{1}{2bi(c^2-b^2)^2}$$

Again, $f(z) = \frac{1}{(z^2+b^2)(z^2+c^2)^2} = \frac{1}{(z^2+b^2)(z-ci)^2(z+ci)^2}$

$$= \frac{\phi(z)}{(z-ci)^2}$$

where, $\phi(z) = \frac{1}{(z^2+b^2)(z+ci)^2}$

$$\therefore \phi'(z) = \frac{-2z}{(z^2+b^2)^2(z+ci)^2} - \frac{2}{(z^2+b^2)(z+ci)^3}$$

Hence, the residue of $f(z)$ at $z = ci$ (twice).

$$= \phi'(ci) = \frac{-2ci}{(b^2-c^2)^2 \times 4c^2 i^2} - \frac{2}{(b^2-c^2) \times 8c^3 i^3}$$

$$= \frac{i}{(b^2-c^2)^2} + \frac{1}{4c^3 i(b^2-c^2)}$$

$$\Sigma R^+ = \frac{1}{2bi(c^2-b^2)^2} + \frac{1}{2(b^2-c^2)^2 c} + \frac{1}{4c^3 i(b^2-c^2)} =$$

$$\frac{2c^3 - 2ibc^2 + b(b^2-c^2)}{4ibc^3(b^2-c^2)^2}$$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \,\Sigma R^+$$

$$= \cancel{2}\pi \cancel{i} \frac{\{2c^3 - 2ibc^2 + b(b^2-c^2)\}}{\underset{2}{\cancel{4}}\cancel{i}\,bc^3(b^2-c^2)^2}$$

$$= \frac{\pi}{2bc^3} \frac{\{2c^2(c-b) + b(b-c)(b+c)\}}{(b-c)^2(b+c)^2}$$

$$= \frac{\pi}{2bc^3} \frac{\{b(b+c) - 2c^2\}}{(b-c)(b+c)^2}$$

$$= \frac{\pi}{2bc^3} \frac{\{(b^2-c^2) + bc - c^2\}}{(b-c)(b+c)^2}$$

$$= \frac{\pi}{2bc^3} \frac{\{(b-c)(b+c) + c(b-c)\}}{(b-c)(b+c)^2}$$

$$= \frac{\pi}{2bc^3} \frac{\cancel{(b-c)}\{(b+c)+c\}}{\cancel{(b-c)}(b+c)^2}$$

$$= \frac{\pi(b+2c)}{2bc^3(b+c)^2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)} = \frac{\pi(b+2c)}{2bc^3(b+c)^2}.$$

Jordan's inequality when $0 \le \theta \le \frac{\pi}{2}$

then, $\frac{2\theta}{\pi} \le \sin\theta \le \theta$.

We have, $\frac{1}{\theta}\int_0^{\theta} \cos x \, dx = \frac{1}{\theta}(\sin x)_0^{\theta}$

$= \frac{\sin\theta}{\theta}$

$\therefore 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} \le \frac{\sin\theta}{\theta} \le 1.$

$\therefore \frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1.$

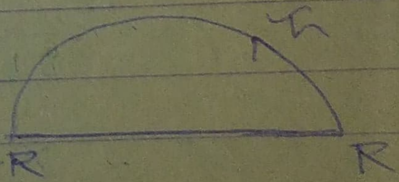
$\therefore \frac{2\theta}{\pi} \le \sin\theta \le \theta.$

QNo> Show that if m & n are +ve integer and $m<n$ then $\int_0^{\infty} \frac{x^{2m}}{x^{2n}+1} dx = \frac{\pi}{2n \sin\frac{2m+1}{2n}\pi}$

Soln:- We consider the integral,

$\int_C \frac{z^{2m}}{1+z^{2n}} dz = \int_C f(z) dz$ where C is contour

~~Consisting~~ Consisting the semi circle



Γ together with real axis with R to -R.

$\therefore \int_C f(z) dz = \int_{-R}^{R} f(x) dx + \int_{\Gamma} f(z) dz = 2\pi i \sum R^{+}$

Since $n > m$.

$\lim_{z\to\infty} z\{f(z)\} = \lim_{z\to\infty} z \cdot \frac{z^{2m}}{1+z^{2n}} = 0$, $2m+1 < 2n$.

$\therefore \lim_{R\to\infty} \int_{\Gamma} f(z) dz = 0 \therefore \int_C f(z) dz = \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum R^{+}$

The Poles of $f(z)$ are given by $z^{2n}+1=0$

i.e. $z^{2n} = -1 = \cos\pi + i\sin\pi$

$= \cos(2r+1)\pi + i\sin(2r+1)\pi$

i.e. $z = \cos(2r+1)\frac{\pi}{2n} + i\sin(2r+1)\frac{\pi}{2n}$

i.e. $e^{(2r+1)\frac{i\pi}{2n}}$ where $r = 0, 1, 2, \dots 2n-1$ of which

only first n' Poles,

i.e. $e^{(2r+1)\frac{i\pi}{2n}}$ where $r = 0, 1, 2, \dots n-1$ lies within e

Let these Poles be $\alpha_1, \alpha_2, \dots \alpha_n$. the residue of

$f(z)$ at $z=\alpha$ will be $\lim_{z\to\alpha} \frac{z^{2m}}{(z^{2n}+1)}$

$= \lim_{z\to\alpha} \frac{z^{2m}}{2nz^{2n-1}} = \frac{\alpha^{2m}}{2n\alpha^{2n-1}}$

$= \frac{\alpha^{2m+1}}{2n\alpha^{2n}} = -\frac{1}{2n}\alpha^{2m+1}$

Since α be root of $z^{2n}+1=0$ therefore

Sum of residue at the Poles $\alpha_1, \alpha_2, \dots \alpha_n$.

$= -\frac{1}{2n}\left(\alpha_1^{2m+1} + \alpha_2^{2m+1} + \dots + \alpha_n^{2m+1}\right)$.

$= -\frac{1}{2n}\left\{e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \dots + e^{i(2n-1)\theta}\right\}$

where, $\theta = \frac{(2m+1)\pi}{2n}$

$= -\frac{1}{2n} \frac{\{e^{i\theta}\{(-e^{i\theta}-1)\}}{1-e^{2i\theta}}$

$= -\frac{1}{2n} \frac{e^{i\theta}\{1-e^{2ni\theta}(1-e^{-2i\theta})}{(1-e^{2i\theta})(1-e^{-2i\theta})}$

$= -\frac{1}{2n} \frac{\{1-e^{2ni\theta} - e^{-2i\theta} + e^{(2n-2)i\theta}\}}{1-(e^{2i\theta}+e^{-2i\theta})+1}$

$$= -\frac{1}{2n}\frac{\{(e^{i\theta}-e^{-i\theta})-e^{(2m+1)i\theta}+e^{(2m-1)i\theta}}{2-2\cos 2\theta}$$

$$= -\frac{1}{4n(1-\cos 2\theta)}\{e^{i\theta}-e^{-i\theta}-e^{2mi\theta}(e^{i\theta}-e^{-i\theta})\}$$

$$= \frac{-\cancel{2}i\sin\theta}{4n\times\cancel{2}\sin^2\theta}(1-e^{2mi\theta}) = \frac{-i}{4n\sin\theta}\{1-\cos 2m\theta - i\sin 2m\theta\}$$

$$= \frac{-i}{4n\sin\theta}(2\sin^2 m\theta - 2i\sin m\theta\cdot\cos m\theta)$$

$$= \frac{-2\sin m\theta}{4n\sin\theta}(\cos m\theta + i\sin m\theta)$$

$$= m\theta = \cancel{n}(2m+1)\frac{\pi}{2\cancel{n}}$$

$$= m + \frac{\pi}{2}$$

$$\therefore \cos m\theta = \cos\left(m\pi + \frac{\pi}{2}\right)$$

$$= (-1)^m \cos\frac{\pi}{2} = 0$$

$$\sin m\theta = \sin\left(m\pi + \frac{\pi}{2}\right) = (-1)^m \sin\frac{\pi}{2} = (-1)^m$$

$$\therefore \Sigma R^+ = \frac{-i\sin^2 m\theta}{2n\sin\theta} = \frac{(-i)(-1)^{2m}}{2n\sin\theta}$$

$$= \frac{-i}{2n\sin\theta}$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\,\Sigma R^+$$

$$= 2\pi i \times \frac{-i}{2n\sin\theta}$$

$$= \frac{\pi}{n\sin\frac{2m+1}{2n}\pi}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{n\sin\frac{2m+1}{2n}\pi}$$

$$\therefore 2\int_{0}^{\infty} \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{n\sin\frac{2m+1}{2n}\pi}$$

$$\therefore \int_{0}^{\infty} \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}$$

QN→ Prove that, $\int_0^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi}{2a} e^{-ma}$ $(m \geq 0)$

deduce that, $\int_0^{\infty} \frac{x \sin mx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-ma}$

Proof:- We Consider the integral

$\int_{-\infty}^{\infty} \frac{e^{imz}}{z^2+a^2}\,dz = \int_C f(z)\,dz$ where C is Contour Consisting of Semi Circle of Γ in upper half the Z Plane together with the real axis from $-R$ to R.

$\therefore \int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$ Since, $\lim_{z\to\infty} \frac{1}{z^2+a^2} = 0$ by Jordan lemma, we have $\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$

$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$ where, $\sum R^+$ is the Some of the resiues at Poles of $f(z)$ within C the Poles of $f(z)$ are given by $z^2+a^2=0$ i.e. $z=\pm ai$ of which only $z=ai$ lies within C the residue of $f(z)$ at $z=ai$ will be given by

$$\lim_{z\to ai} (z-ai)f(z) = \lim_{z\to ai} \frac{(z-ai)e^{imz}}{z^2+a^2} = \lim_{z\to ai} \frac{e^{imz}}{z+ai} = \frac{e^{im}}{2a}$$

$$= \frac{e^{-ma}}{2ai}$$

$\therefore \int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx = 2\pi i \Sigma R^+$

$= 2\pi i \dfrac{e^{-ma}}{2ia} = \dfrac{\pi e^{-ma}}{a} = \int_{-\infty}^{\infty} \dfrac{e^{imx}}{x^2+a^2} dx = \int_{-\infty}^{\infty} \dfrac{\cos mx + i\sin mx}{x^2+a^2} dx$

$= \dfrac{\pi}{a} e^{-ma}$

Equating real parts from both sides, we have

$\int_{-\infty}^{\infty} \dfrac{\cos mx}{x^2+a^2} dx = \dfrac{\pi}{a} e^{-ma} \therefore 2\int_0^{\infty} \dfrac{\cos mx}{x^2+a^2} = \dfrac{\pi}{a} e^{-ma}.$

$\therefore \int_0^{\infty} \dfrac{\cos mx}{x^2+a^2} dx = \dfrac{\pi}{2a} e^{-ma}.$

Diff with the respect to m, we have

$\int_0^{\infty} \dfrac{-x \sin mx}{x^2+a^2} = \dfrac{\pi}{2} e^{-ma}.$

M.U. M.Sc 04

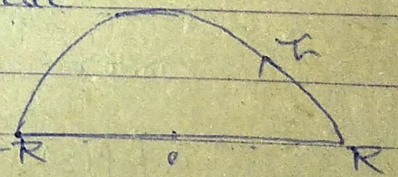
Q No → Prove by Contour integration that,

$$\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2.$$

Proof:- Let us Consider the integral

$$\int_C \frac{\log(z+i)}{z^2+1} dz = \int_C f(z)dz$$

Where C is Contour Consisting of the real axis from $-R$ to R and the semi circle Γ of large radius R in the upper half of the z plane.



$\therefore \int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz = 2\pi i \Sigma R^+.$ Where, ΣR^+ is the sum of the residues of $f(z)$ at poles of $f(z)$ within C, we have

$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \dfrac{z \cdot \log(z+i)}{z^2+1} = 0$

$\therefore \lim_{R\to\infty} \int_\Gamma f(z)dz = 0 \therefore \int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx = 2\pi i \Sigma R^+.$

the Poles of $f(z)$ are given by $z^2 = -1 = i^2$ of which only lies within C, the residue of $f(z)$ at $z = i$ will be given by

$$\lim_{z \to i} (z-i) f(z) = \lim_{z \to i} \frac{(z-i)\log(z+i)}{z^2+1}$$

$$= \lim_{z \to i} \frac{\log(z+i)}{z+i} = \frac{\log 2i}{2i} = \frac{\log 2 + \log i}{2i}$$

$$= \frac{\log 2 + \log e^{i\pi/2}}{2i} = \frac{\log 2 + i\pi/2}{2i}$$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \Sigma R^+$$

$$= \frac{2\pi i\left(\log 2 + \frac{i\pi}{2}\right)}{2i} = \pi\left(\log 2 + \frac{i\pi}{2}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1}\,dx = \pi\left(\log 2 + \frac{i\pi}{2}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(x^2+1) + i\tan^{-1}\frac{1}{x}}{x^2+1}\,dx$$

$$= \pi \log 2 + \frac{i\pi^2}{2}$$

Equating the real Part from both Sides, we have

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx = \pi \log 2 \quad \therefore \frac{1}{\cancel{2}} \cdot \cancel{2} \int_0^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx = \pi \log 2$$